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# Split feasibility problems for total quasi-asymptotically nonexpansive mappings

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## Abstract

The purpose of this paper is to propose an algorithm for solving the *split feasibility problems* for *total quasi-asymptotically nonexpansive mappings* in infinite-dimensional Hilbert spaces. The results presented in the paper not only improve and extend some recent results of Moudafi [Nonlinear Anal. 74:4083-4087, 2011; Inverse Problem 26:055007, 2010], but also improve and extend some recent results of Xu [Inverse Problems 26:105018, 2010; 22:2021-2034, 2006], Censor and Segal [J. Convex Anal. 16:587-600, 2009], Censor *et al.* [Inverse Problems 21:2071-2084, 2005], Masad and Reich [J. Nonlinear Convex Anal. 8:367-371, 2007], Censor *et al.* [J. Math. Anal. Appl. 327:1244-1256, 2007], Yang [Inverse Problem 20:1261-1266, 2004] and others.

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**Keywords:** split feasibility problem; convex feasibility problem; total quasi-asymptotically nonexpansive mappings; demi-closeness; Opial condition

## 1 Introduction

Throughout this paper, we always assume that $H_1$, $H_2$ are real Hilbert spaces, '$\to$', '$\rightharpoonup$' denote strong and weak convergence, respectively, and $F(T)$ is a fixed point set of a mapping $T$.

The *split feasibility problem* (SFP) in finite-dimensional spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. Recently, it has been found that the SFP can also be used in various disciplines such as image restoration, computer tomograph and radiation therapy treatment planning [3–5]. The *split feasibility problem* in an infinite-dimensional real Hilbert space can be found in [2, 4, 6–10].

The purpose of this paper is to introduce and study the following *split feasibility problem* for *total quasi-asymptotically nonexpansive mappings* in the framework of infinite-dimensional real Hilbert spaces:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are mappings; $C := F(S)$ and $Q := F(T)$. In the sequel, we use $\Gamma$ to denote the set of solutions of

(SFP)-(1.1), *i.e.*,

$$\Gamma = \{x \in C, Ax \in Q\}. \tag{1.2}$$

## 2 Preliminaries

We first recall some definitions, notations and conclusions which will be needed in proving our main results.

Let $E$ be a Banach space. A mapping $T : E \to E$ is said to be *demi-closed at origin* if for any sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x^*$ and $\|(I - T)x_n\| \to 0$, $x^* = Tx^*$.

A Banach space $E$ is said to have *the Opial property*, if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x^*$,

$$\liminf_{n\to\infty} \|x_n - x^*\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \text{ with } y \neq x^*.$$

**Remark 2.1** It is well known that each Hilbert space possesses the Opial property.

**Definition 2.2** Let $H$ be a real Hilbert space.

(1) A mapping $G : H \to H$ is said to be a $(\{\nu_n\}, \{\mu_n\}, \zeta)$*-total quasi-asymptotically nonexpansive mapping* if $F(G) \neq \emptyset$; and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$ and $\mu_n \to 0$ and a strictly increasing continuous function $\zeta : \mathcal{R}^+ \to \mathcal{R}^+$ with $\zeta(0) = 0$ such that for each $n \geq 1$,

$$\|p - G^n x\|^2 \leq \|p - x\|^2 + \nu_n \zeta(\|p - x\|) + \mu_n, \quad \forall p \in F(G), x \in H. \tag{2.1}$$

Now, we give an example of total quasi-asymptotically nonexpansive mapping.

Let $C$ be a unit ball in a real Hilbert space $l^2$, and let $T : C \to C$ be a mapping defined by

$$T : (x_1, x_2, \ldots,) \to (0, x_1^2, a_2 x_2, a_3 x_3, \ldots), (x_1, x_2, \ldots,) \in l^2,$$

where $\{a_i\}$ is a sequence in $(0, 1)$ such that $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$.

It is proved in Goebal and Kirk [17] that

(i) $\|Tx - Ty\| \leq 2\|x - y\|$, $\forall x, y \in C$;

(ii) $\|T^n x - T^n y\| \leq 2 \prod_{j=2}^{n} a_j \|x - y\|$, $\forall x, y \in C$, $\forall n \geq 2$.

Denote by $k_1^{\frac{1}{2}} = 2$, $k_n^{\frac{1}{2}} = 2 \prod_{j=2}^{n} a_j$, $n \geq 2$, then

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} \left(2 \prod_{j=2}^{n} a_j\right)^2 = 1.$$

Letting $\nu_n = (k_n - 1)$, $\forall n \geq 1$, $\zeta(t) = t$, $\forall t \geq 0$ and $\{\mu_n\}$ be a nonnegative real sequence with $\mu_n \to 0$, from (i) and (ii), $\forall x, y \in C$, $n \geq 1$, we have

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + \nu_n \zeta(\|x - y\|^2) + \mu_n. \tag{2.2}$$

Again, since $0 \in C$ and $0 \in F(T)$, this implies that $F(T) \neq \emptyset$. From (2.2), we have

$$\|p - T^n y\|^2 \leq \|p - y\|^2 + \nu_n \zeta(\|p - y\|^2) + \mu_n \quad \forall p \in F(T), y \in C. \tag{2.3}$$

This shows that the mapping $T$ defined as above is a total quasi-asymptotically nonexpansive mapping.

(2) A mapping $G : H \to H$ is said to be $(\{k_n\})$-*quasi-asymptotically nonexpansive* if $F(G) \neq \emptyset$; and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for all $n \geq 1$,

$$\|p - G^n x\|^2 \leq k_n \|p - x\|^2, \quad \forall p \in F(G), x \in H. \tag{2.4}$$

(3) A mapping $G : H \to H$ is said to be *quasi-nonexpansive* if $F(G) \neq \emptyset$ such that

$$\|p - Gx\| \leq \|p - x\|, \quad \forall p \in F(G), x \in H. \tag{2.5}$$

**Remark 2.3** It is easy to see that every quasi-nonexpansive mapping is a $(\{1\})$-quasi-asymptotically nonexpansive mapping and every $\{k_n\}$-quasi-asymptotically nonexpansive mapping is a $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total quasi-asymptotically nonexpansive mapping with $\{\nu_n = k_n - 1\}$, $\{\mu_n = 0\}$ and $\zeta(t) = t^2$, $t \geq 0$.

**Definition 2.4**

(1) A mapping $G : H \to H$ is said to be uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall x, y \in H \text{ and } n \geq 1.$$

(2) A mapping $G : H \to H$ is said to be *semi-compact* if for any bounded sequence $\{x_n\} \subset H$ with $\lim_{n\to\infty} \|x_n - Gx_n\| = 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i}$ converges strongly to some point $x^* \in H$.

**Proposition 2.5** *Let $G : H \to H$ be a $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total quasi-asymptotically nonexpansive mapping. Then for each $q \in F(G)$ and for each $x \in H$, the following inequalities are equivalent: for each $n \geq 1$*

$$\|q - G^n x\|^2 \leq \|q - x\|^2 + \nu_n \zeta\left(\|q - x\|\right) + \mu_n, \quad \forall q \in F(G), x \in H; \tag{2.1}$$

$$2\langle x - G^n x, x - q\rangle \geq \|x - G^n x\|^2 - \nu_n \zeta\left(\|q - x\|\right) - \mu_n; \tag{2.6}$$

$$2\langle x - G^n x, q - G^n x\rangle \leq \|x - G^n x\|^2 + \nu_n \zeta\left(\|q - x\|\right) + \mu_n. \tag{2.7}$$

*Proof*

(I) (2.1) $\Leftrightarrow$ (2.6) In fact, since

$$\begin{aligned} \|G^n x - q\|^2 &= \|G^n x - x + x - q\|^2 \\ &= \|G^n x - x\|^2 + \|x - q\|^2 + 2\langle G^n x - x, x - q\rangle, \quad \forall x \in H, q \in F(G), \end{aligned}$$

from (2.1) we have that

$$\begin{aligned} &\|G^n x - x\|^2 + \|x - q\|^2 + 2\langle G^n x - x, x - q\rangle \\ &\quad \leq \|x - q\|^2 + \nu_n \zeta\left(\|q - x\|\right) + \mu_n. \end{aligned}$$

Simplifying it, inequality (2.6) is obtained.

Conversely, from (2.6) the inequality (2.1) can be obtained immediately.

(II) (2.6) $\Leftrightarrow$ (2.7) In fact, since

$$\begin{aligned}\langle x - G^n x, x - q\rangle &= \langle x - G^n x, x - G^n x + G^n x - q\rangle \\ &= \|x - G^n x\|^2 + \langle x - G^n x, G^n x - q\rangle\end{aligned}$$

it follows from (2.6) that

$$2\left(\|x - G^n x\|^2 + \langle x - G^n x, G^n x - q\rangle\right) \geq \|x - G^n x\|^2 - \nu_n \zeta\left(\|q - x\|\right) - \mu_n.$$

Simplifying it, the inequality (2.7) is obtained.

Conversely, from (2.7) the inequality (2.6) can be obtained immediately.

This completes the proof of Proposition 2.5. □

**Lemma 2.6** [11] *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n \geq 1.$$

*If $\sum_{i=1}^{\infty} \delta_n < \infty$ and $\sum_{i=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists.*

## 3 Split feasibility problem

For solving the split feasibility problem (1.1), let us assume that the following conditions are satisfied:

1. $H_1$ and $H_2$ are two real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator;
2. $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are two uniformly $L$-Lipschitzian and $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total quasi-asymptotically nonexpansive mappings satisfying the following conditions:
   (i) $T$ and $S$ both are demi-closed at origin;
   (ii) $\sum_{n=1}^{\infty}(\mu_n + \nu_n) < \infty$;
   (iii) there exist positive constants $M$ and $M^*$ such that $\zeta(t) \leq \zeta(M) + M^* t^2$, $\forall t \geq 0$.

We are now in a position to give the following result.

**Theorem 3.1** *Let $H_1$, $H_2$, $A$, $S$, $T$, $L$, $\{\mu_n\}$, $\{\nu_n\}$, $\zeta$ be the same as above. Let $\{x_n\}$ be the sequence generated by:*

$$\begin{cases} x_1 \in H_1 \quad \text{chosen arbitrarily}, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n S^n(u_n), \\ u_n = x_n + \gamma A^*(T^n - I)Ax_n, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

*where $\{\alpha_n\}$ is a sequence in $[0,1]$, and $\gamma > 0$ is a constant satisfying the following conditions:*

*(iv)* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$*; and* $\gamma \in (0, \frac{1}{\|A\|^2})$,

*(I) If $\Gamma \neq \emptyset$ (where $\Gamma$ is the set of solutions to ((SFP)-(1.1)), then $\{x_n\}$ converges weakly to a point $x^* \in \Gamma$.*

*(II) In addition, if $S$ is also semi-compact, then $\{x_n\}$ and $\{u_n\}$ both converge strongly to $x^* \in \Gamma$.*

*The proof of conclusion (I)*

(1) First, we prove that for each $p \in \Gamma$, the following limits exist:

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|u_n - p\|. \tag{3.2}$$

In fact, since $p \in \Gamma$, we have $p \in C := F(S)$ and $Ap \in Q := F(T)$. It follows from (3.1) and (2.4) that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \big\|u_n - p - \alpha_n\big(u_n - S^n u_n\big)\big\|^2 \\ &= \|u_n - p\|^2 - 2\alpha_n\langle u_n - p, u_n - S^n u_n\rangle + \alpha_n^2\big\|u_n - S^n u_n\big\|^2 \\ &\le \|u_n - p\|^2 - \alpha_n\big\{\big\|u_n - S^n u_n\big\|^2 - \nu_n\zeta\big(\|u_n - p\|\big) - \mu_n\big\} \\ &\quad + \alpha_n^2\big\|u_n - S^n u_n\big\|^2 \quad \big(\text{by (2.6)}\big) \\ &= \|u_n - p\|^2 - \alpha_n(1-\alpha_n)\big\|u_n - S^n u_n\big\|^2 + \alpha_n\big(\nu_n\zeta\big(\|u_n - p\|\big) + \mu_n\big). \end{aligned} \tag{3.3}$$

On the other hand, by condition (iii), we have

$$\zeta\big(\|u_n - p\|\big) \le \zeta(M) + M^*\|u_n - p\|^2. \tag{3.4}$$

Substituting (3.4) into (3.3) and simplifying, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \big(1 + \alpha_n\nu_n M^*\big)\|u_n - p\|^2 - \alpha_n(1-\alpha_n)\big\|u_n - S^n u_n\big\|^2 \\ &\quad + \alpha_n\big(\nu_n\zeta(M) + \mu_n\big) \\ &\le \big(1 + \nu_n M^*\big)\|u_n - p\|^2 - \alpha_n(1-\alpha_n)\big\|u_n - S^n u_n\big\|^2 + \nu_n\zeta(M) + \mu_n. \end{aligned} \tag{3.5}$$

On the other hand,

$$\begin{aligned} \|u_n - p\|^2 &= \big\|x_n - p + \gamma A^*\big(T^n - I\big)Ax_n\big\|^2 \\ &= \|x_n - p\|^2 + \gamma^2\big\|A^*\big(T^n - I\big)Ax_n\big\|^2 + 2\gamma\big\langle x_n - p, A^*\big(T^n - I\big)Ax_n\big\rangle, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \gamma^2\big\|A^*\big(T^n - I\big)Ax_n\big\|^2 &= \gamma^2\big\langle A^*\big(T^n - I\big)Ax_n, A^*\big(T^n - I\big)Ax_n\big\rangle \\ &= \gamma^2\big\langle AA^*\big(T^n - I\big)Ax_n, \big(T^n - I\big)Ax_n\big\rangle \\ &\le \gamma^2\|A\|^2\big\|T^n Ax_n - Ax_n\big\|^2, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &2\gamma\big\langle x_n - p, A^*\big(T^n - I\big)Ax_n\big\rangle \\ &\quad = 2\gamma\big\langle Ax_n - Ap, \big(T^n - I\big)Ax_n\big\rangle \\ &\quad = 2\gamma\big\langle Ax_n - Ap + \big(T^n - I\big)Ax_n - \big(T^n - I\big)Ax_n, \big(T^n - I\big)Ax_n\big\rangle \\ &\quad = 2\gamma\big\{\big\langle T^n Ax_n - Ap, T^n Ax_n - Ax_n\big\rangle - \big\|\big(T^n - I\big)Ax_n\big\|^2\big\}. \end{aligned} \tag{3.8}$$

In (2.5), taking $x = Ax_n$, $G^n = T^n$, $q = Ap$, and noting $Ap \in F(T)$, from (2.7) and condition (iii), we have

$$\begin{aligned}&\langle T^n Ax_n - Ap, T^n Ax_n - Ax_n\rangle \\ &\quad\le \frac{1}{2}\{\|(T^n - I)Ax_n\|^2 + \nu_n\zeta(\|Ax_n - Ap\|) + \mu_n\} \\ &\quad\le \frac{1}{2}\{\|(T^n - I)Ax_n\|^2 + \nu_n(\zeta(M) + M^*\|A\|^2\|x_n - p\|^2) + \mu_n\}.\end{aligned}\tag{3.9}$$

Substituting (3.9) into (3.8) and simplifying it, we have

$$\begin{aligned}&2\gamma\langle x_n - p, A^*(T^n - I)Ax_n\rangle \\ &\quad\le \gamma\{\nu_n(\zeta(M) + M^*\|A\|^2\|x_n - p\|^2) + \mu_n - \|(T^n - I)Ax_n\|^2\}.\end{aligned}\tag{3.10}$$

Substituting (3.7) and (3.10) into (3.6) after simplifying, we have

$$\begin{aligned}\|u_n - p\|^2 \le{}& (1 + \gamma\nu_n M^*\|A\|^2)\|x_n - p\|^2 + \gamma(\nu_n\zeta(M) + \mu_n) \\ &- \gamma(1 - \gamma\|A\|^2)\|(T^n - I)Ax_n\|^2.\end{aligned}\tag{3.11}$$

Substituting (3.11) into (3.5) and simplifying it, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 \le{}& (1 + \nu_n M^*)\{(1 + \gamma\nu_n M^*\|A\|^2)\|x_n - p\|^2 \\ &+ \gamma(\nu_n\zeta(M) + \mu_n) - \gamma(1 - \gamma\|A\|^2)\|(T^n - I)Ax_n\|^2\} \\ &- \alpha_n(1 - \alpha_n)\|u_n - S^n u_n\|^2 + \nu_n\zeta(M) + \mu_n \\ \le{}& (1 + \xi_n)\|x_n - p\|^2 + \eta_n - \gamma(1 - \gamma\|A\|^2)\|(T^n - I)Ax_n\|^2 \\ &- \alpha_n(1 - \alpha_n)\|u_n - S^n u_n\|^2,\end{aligned}\tag{3.12}$$

where

$$\begin{aligned}\xi_n &= \nu_n(M^* + \gamma M^*\|A\|^2 + \gamma\nu_n M^*\|A\|^2), \\ \eta_n &= [(1 + \nu_n M^*)\gamma + 1](\nu_n\zeta(M) + \mu_n).\end{aligned}$$

By condition (iii), we have

$$\sum_{n=1}^{\infty}\xi_n < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty}\eta_n < \infty.$$

By condition (iv), $(1 - \gamma\|A\|^2) > 0$. Hence, from (3.12), we have

$$\|x_{n+1} - p\|^2 \le (1 + \xi_n)\|x_n - p\|^2 + \eta_n, \quad \forall n \ge 1.$$

By Lemma 2.6, the following limit exists:

$$\lim_{n\to\infty}\|x_n - p\|.\tag{3.13}$$

Now, we rewrite (3.12) as follows:

$$\begin{aligned}
&\gamma\left(1-\gamma\|A\|^2\right)\left\|\left(T^n-I\right)Ax_n\right\|^2+\alpha_n(1-\alpha_n)\left\|u_n-S^nu_n\right\|^2\\
&\quad\le\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\qquad+\xi_n\|x_n-p\|^2+\eta_n\to 0\quad(\text{as } n\to\infty).
\end{aligned}$$

This together with the condition (iv) implies that

$$\lim_{n\to\infty}\left\|u_n-S^nu_n\right\|=0; \tag{3.14}$$

and

$$\lim_{n\to\infty}\left\|\left(T^n-I\right)Ax_n\right\|=0. \tag{3.15}$$

It follows from (3.6), (3.14) and (3.15) that the limit $\lim_{n\to\infty}\|u_n-p\|$ exists and

$$\lim_{n\to\infty}\|u_n-p\|=\lim_{n\to\infty}\|x_n-p\|.$$

The conclusion (3.2) is proved.

(2) Next, we prove that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0\quad\text{and}\quad\lim_{n\to\infty}\|u_{n+1}-u_n\|=0. \tag{3.16}$$

In fact, it follows from (3.1) that

$$\begin{aligned}
\|x_{n+1}-x_n\|&=\left\|(1-\alpha_n)u_n+\alpha_nS^n(u_n)-x_n\right\|\\
&=\left\|(1-\alpha_n)\left(x_n+\gamma A^*\left(T^n-I\right)Ax_n\right)+\alpha_nS^n(u_n)-x_n\right\|\\
&=\left\|(1-\alpha_n)\gamma A^*\left(T^n-I\right)Ax_n+\alpha_n\left(S^n(u_n)-x_n\right)\right\|\\
&=\left\|(1-\alpha_n)\gamma A^*\left(T^n-I\right)Ax_n+\alpha_n\left(S^n(u_n)-u_n\right)+\alpha_n(u_n-x_n)\right\|\\
&=\left\|(1-\alpha_n)\gamma A^*\left(T^n-I\right)Ax_n+\alpha_n\left(S^n(u_n)-u_n\right)+\alpha_n\gamma A^*\left(T^n-I\right)Ax_n\right\|\\
&=\left\|\gamma A^*\left(T^n-I\right)Ax_n+\alpha_n\left(S^n(u_n)-u_n\right)\right\|.
\end{aligned}$$

In view of (3.14) and (3.15), we have that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.17}$$

Similarly, it follows from (3.1), (3.15) and (3.17) that

$$\begin{aligned}
\|u_{n+1}-u_n\|&=\left\|x_{n+1}+\gamma A^*\left(T^{n+1}-I\right)Ax_{n+1}-\left(x_n+\gamma A^*\left(T^n-I\right)Ax_n\right)\right\|\\
&\le\|x_{n+1}-x_n\|+\gamma\left\|A^*\left(T^{n+1}-I\right)Ax_{n+1}\right\|\\
&\quad+\gamma\left\|A^*\left(T^n-I\right)Ax_n\right\|\to 0\quad(\text{as } n\to\infty).
\end{aligned} \tag{3.18}$$

The conclusion (3.16) is proved.

(3) Next, we prove that

$$\|u_n - Su_n\| \to 0 \quad \text{and} \quad \|Ax_n - TAx_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.19}$$

In fact, from (3.14), we have

$$\zeta_n := \left\|u_n - S^n u_n\right\| \to 0 \quad (\text{as } n \to \infty). \tag{3.20}$$

Since $S$ is uniformly $L$-Lipschitzian continuous, it follows from (3.16) and (3.20) that

$$\begin{aligned}
\|u_n - Su_n\| &\le \left\|u_n - S^n u_n\right\| + \left\|S^n u_n - Su_n\right\| \\
&\le \zeta_n + L\left\|S^{n-1}u_n - u_n\right\| \\
&\le \zeta_n + L\left\{\left\|S^{n-1}u_n - S^{n-1}u_{n-1}\right\| \right.\\
&\quad \left. + \left\|S^{n-1}u_{n-1} - u_n\right\|\right\} \\
&\le \zeta_n + L^2\|u_n - u_{n-1}\| \\
&\quad + L\left\|S^{n-1}u_{n-1} - u_{n-1} + u_{n-1} - u_n\right\| \\
&\le \zeta_n + L(1+L)\|u_n - u_{n-1}\| + L\zeta_{n-1} \to 0 \quad (\text{as } n \to \infty).
\end{aligned}$$

Similarly, from (3.15), we have

$$\left\|Ax_n - T^n Ax_n\right\| \to 0 \quad (\text{as } n \to \infty). \tag{3.21}$$

Since $T$ is uniformly $L$-Lipschitzian continuous, by the same way as above, from (3.16) and (3.21), we can also prove that

$$\|Ax_n - TAx_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.22}$$

(4) Finally, we prove that $x_n \rightharpoonup x^*$ and $u_n \rightharpoonup x^*$, which is a solution of (SFP)-(1.1).

Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\} \subset \{u_n\}$ such that $u_{n_i} \rightharpoonup x^*$ (some point in $H_1$). From (3.19), we have

$$\|u_{n_i} - Su_{n_i}\| \to 0 \quad (\text{as } n_i \to \infty). \tag{3.23}$$

By the assumption that $S$ is demi-closed at zero, we get that $x^* \in F(S)$.

Moreover, from (3.1) and (3.15), we have

$$x_{n_i} = u_{n_i} - \gamma A^*\left(T^{n_i} - I\right)Ax_{n_i} \rightharpoonup x^*.$$

Since $A$ is a linear bounded operator, we get $Ax_{n_i} \rightharpoonup Ax^*$. In view of (3.19), we have

$$\|Ax_{n_i} - TAx_{n_i}\| \to 0 \quad (\text{as } n_i \to \infty).$$

Since $T$ is demi-closed at zero, we have $Ax^* \in F(T)$. Summing up the above argument, it is clear that $x^* \in \Gamma$, *i.e.*, $x^*$ is a solution to the (SFP)-(1.1).

Now, we prove that $x_n \rightharpoonup x^*$ and $u_n \rightharpoonup x^*$.

Suppose, to the contrary, that if there exists another subsequence $\{u_{n_j}\} \subset \{u_n\}$ such that $u_{n_j} \rightharpoonup y^* \in \Gamma$ with $y^* \neq x^*$, then by virtue of (3.2) and the Opial property of Hilbert space, we have

$$\begin{aligned}\liminf_{n_i\to\infty}\|u_{n_i}-x^*\| &< \liminf_{n_i\to\infty}\|u_{n_i}-y^*\| = \lim_{n\to\infty}\|u_n-y^*\|\\ &= \lim_{n_j\to\infty}\|u_{n_j}-y^*\| < \liminf_{n_j\to\infty}\|u_{n_j}-x^*\|\\ &= \lim_{n\to\infty}\|u_n-x^*\| = \liminf_{n_i\to\infty}\|u_{n_i}-x^*\|.\end{aligned}$$

This is a contradiction. Therefore, $u_n \rightharpoonup x^*$. By using (3.1) and (3.15), we have

$$x_n = u_n - \gamma A^*(T_n^n - I)Ax_n \rightharpoonup x^*. \qquad \square$$

*The proof of conclusion (II)* By the assumption that $S$ is semi-compact, it follows from (3.23) that there exists a subsequence of $\{u_{n_i}\}$ (without loss of generality, we still denote it by $\{u_{n_i}\}$) such that $u_{n_i} \to u^* \in H$ (some point in $H$). Since $u_{n_i} \rightharpoonup x^*$. This implies that $x^* = u^*$, and so $u_{n_i} \to x^* \in \Gamma$. By virtue of (3.2), we know that $\lim_{n\to\infty}\|u_n - x^*\| = 0$ and $\lim_{n\to\infty}\|x_n - x^*\| = 0$, *i.e.*, $\{u_n\}$ and $\{x_n\}$ both converge strongly to $x^* \in \Gamma$.

This completes the proof of Theorem 3.1. $\square$

**Theorem 3.2** *Let $H_1$, $H_2$ and $A$ be the same as in Theorem 3.1. Let $S: H_1 \to H_1$ and $T: H_2 \to H_2$ be two $(\{k_n\})$-quasi-asymptotically nonexpansive mappings with $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ satisfying the following conditions:*

*(i) $T$ and $S$ both are demi-closed at origin;*

*(ii) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$.*

*Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in H_1 \quad \textit{chosen arbitrarily}, \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n S^n(u_n), \\ u_n = x_n + \gamma A^*(T^n - I)Ax_n, \quad \forall n \geq 1, \end{cases} \tag{3.24}$$

*where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\gamma > 0$ is a constant satisfying the condition (iv) in Theorem 3.1. Then the conclusions in Theorem 3.1 still hold.*

*Proof* By assumptions, $S: H_1 \to H_1$ and $T: H_2 \to H_2$ both are $(\{k_n\})$-quasi-asymptotically nonexpansive mappings with $\{k_n\} \subset [1,\infty)$, $k_n \to 1$; by Remark 2.3, $S$ and $T$ both are uniformly $L$-Lipschitzian (where $L = \sup_{n\geq 1} k_n$) and $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total quasi-asymptotically nonexpansive mapping with $\{\nu_n = k_n - 1\}$, $\{\mu_n = 0\}$ and $\zeta(t) = t^2$, $t \geq 0$. Therefore, all conditions in Theorem 3.1 are satisfied. The conclusions of Theorem 3.2 can be obtained from Theorem 3.1 immediately. $\square$

**Theorem 3.3** *Let $H_1$, $H_2$ and $A$ be the same as in Theorem 3.1. Let $S: H_1 \to H_1$ and $T: H_2 \to H_2$ be two quasi-nonexpansive mappings and demi-closed at origin. Let $\{x_n\}$ be the* *sequence generated by*

$$\begin{cases} x_1 \in H_1 \quad \textit{chosen arbitrarily}, \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n S^n(u_n), \\ u_n = x_n + \gamma A^*(T^n - I)Ax_n, \quad \forall n \geq 1, \end{cases} \tag{3.25}$$

*where* $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *and* $\gamma > 0$ *is a constant satisfying the condition (iv) in Theorem 3.1. Then the conclusions in Theorem 3.1 still hold.*

*Proof* By the assumptions, $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are quasi-nonexpansive mappings. By Remark 2.3, $S$ and $T$ both are uniformly $L$-Lipschitzian (where $L = 1$) and ($\{1\}$)-quasi-asymptotically nonexpansive mappings. Therefore, all conditions in Theorem 3.2 are satisfied. The conclusions of Theorem 3.3 can be obtained from Theorem 3.2 immediately. □

**Remark 3.4** Theorems 3.1, 3.2 and 3.3 not only improve and extend the corresponding results of Moudafi [12, 13], but also improve and extend the corresponding results of Censor *et al.* [4, 5], Yang [7], Xu [14], Censor and Segal [15], Masad and Reich [16] and others.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed to this work equal. All authors read and ap- proved the final manuscript.

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